3-D GEOMETRY: A TRIANGLE-ORIENTED PROOF OF THE COSINE OF THE ANGLE BETWEEN TWO VECTORS

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ABSTRACT

The well-established formula for the cosine of the angle between two vectors in 3-dimensional space is established using straightforward results of triangle geometry.

KEY WORDS

Cartesian coordinates, right-angle triangles, rotation of axes.

INTRODUCTION

Preparation of a brief article for an encyclopedia on the cosine of the angle between two vectors led first to locating the results as a definition. For vectors $\mathbf{r}$ and $\mathbf{s}$ in n-space, in which $\mathbf{r} = [r_1 \ \ r_2 \cdots r_i \cdots r_n]'$ and $\mathbf{s} = [s_1 \ \ s_2 \cdots s_i \cdots s_n]'$ are points, respectively, we find in Johnson and Wichern (1988, p. 69) that the angle θ between $\mathbf{r}$ and $\mathbf{s}$ is defined by

$$\cos\theta = \mathbf{r}'\mathbf{s}/\sqrt{\mathbf{r}'\mathbf{r}(\mathbf{s}'\mathbf{s})} \ . \tag{1}$$

Geometrically this may not seem very satisfactory because although two vectors in 2-space always intersect (unless they are parallel), two vectors in 3-space may not; and (1) seems devoid of any requirement that vectors $\mathbf{r}$ and $\mathbf{s}$ intersect (except, of course, that even in n-space it is often assumed that all vectors start at the origin of the coordinates). Thus, as a *definition* of an angle, (1) may be satisfactory although not necessarily of an angle in the usual 2-dimensional sense. This prompts consideration of (1) for 2-space and 3-space. The former is easy, whereas the latter is a little more difficult; it is given in McCrea (1947, p. 9) in terms of projections, but it can also be developed in terms of triangle geometry, which is the purpose of this note.

2-SPACE

Figure 1 shows coordinates x_1,y_1 and x_2,y_2 of points of two vectors in 2-space.

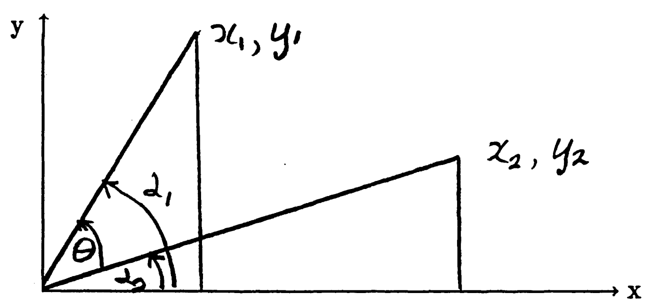

Figure 1. Two vectors in 2-space.

Dropping perpendiculars from those points on to the x-axis, as shown, easily gives

$$\begin{aligned}\cos\theta &= \cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2 \\ &= \frac{x_1x_2}{d_1d_2} + \frac{y_1y_2}{d_1d_2} = \frac{x_1x_2 + y_1y_2}{d_1d_2} \ ,\end{aligned} \tag{2}$$

where $d_i = \sqrt{x_i^2 + y_i^2}$ for i = 1, 2. And (2) is the 2-dimensional form of (1).

3-SPACE

The diagram for 3-space is Figure 2 (see page 4). It is comparable to Figure 1 but a little more complicated. 0 is the origin of the Cartesian coordinates x, y and z. P_1 and P_2 with coordinates (x_1,y_1,z_1) and (x_2,y_2,z_2), respectively, are two points, with $0P_1$ and $0P_2$ being the vectors, intersecting at 0, and thus defining a 2-dimensional plane. θ, the angle of interest, is the angle between the vectors in that plane. We show that

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{d_1d_2}$$

with

$$d_i = \sqrt{x_i^2 + y_i^2 + z_i^2} \quad \text{for} \quad i = 1 \text{ and } 2\,. \tag{3}$$

In Figure 2, perpendiculars to the x,y plane through P_1 and P_2 meet the plane at R_1 and R_2, respectively. A plane through P_2 parallel to the x,y plane is intersected by P_1R_1 at S. $SQ_2P_2Q_1$ is a rectangle in that plane.

We derive $\cos\theta$ by finding the length of each side of the triangle P_10P_2. First, from the rectangle having diagonal $0R_1$ its length $|0R_1|$ is given by

$$|0R_1|^2 = x_1^2 + y_1^2\,.$$

Therefore from the triangle OR_1P_1

$$|0P_1|^2 = (x_1^2 + y_1^2) + z_1^2 = x_1^2 + y_1^2 + z_1^2 = d_1^2\,.$$

Similarly

$$|0P_2|^2 = x_2^2 + y_2^2 + z_2^2 = d_2^2\,.$$

Consideration of the rectangle $SQ_2P_2Q_1$ reveals that

$$|SQ_1| = x_2 - x_1, \qquad |SQ_2| = y_2 - y_1$$

and clearly

$$|SP_1| = z_2 - z_1\,.$$

Therefore

$$|SP_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and so

$$\begin{aligned} |P_1P_2|^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= w^2, \text{ say}\,. \end{aligned} \tag{4}$$

We now have $\Delta\, P_10P_2$ having sides d_1,d_2 and w, in which we drop a perpendicular from P_1 on to

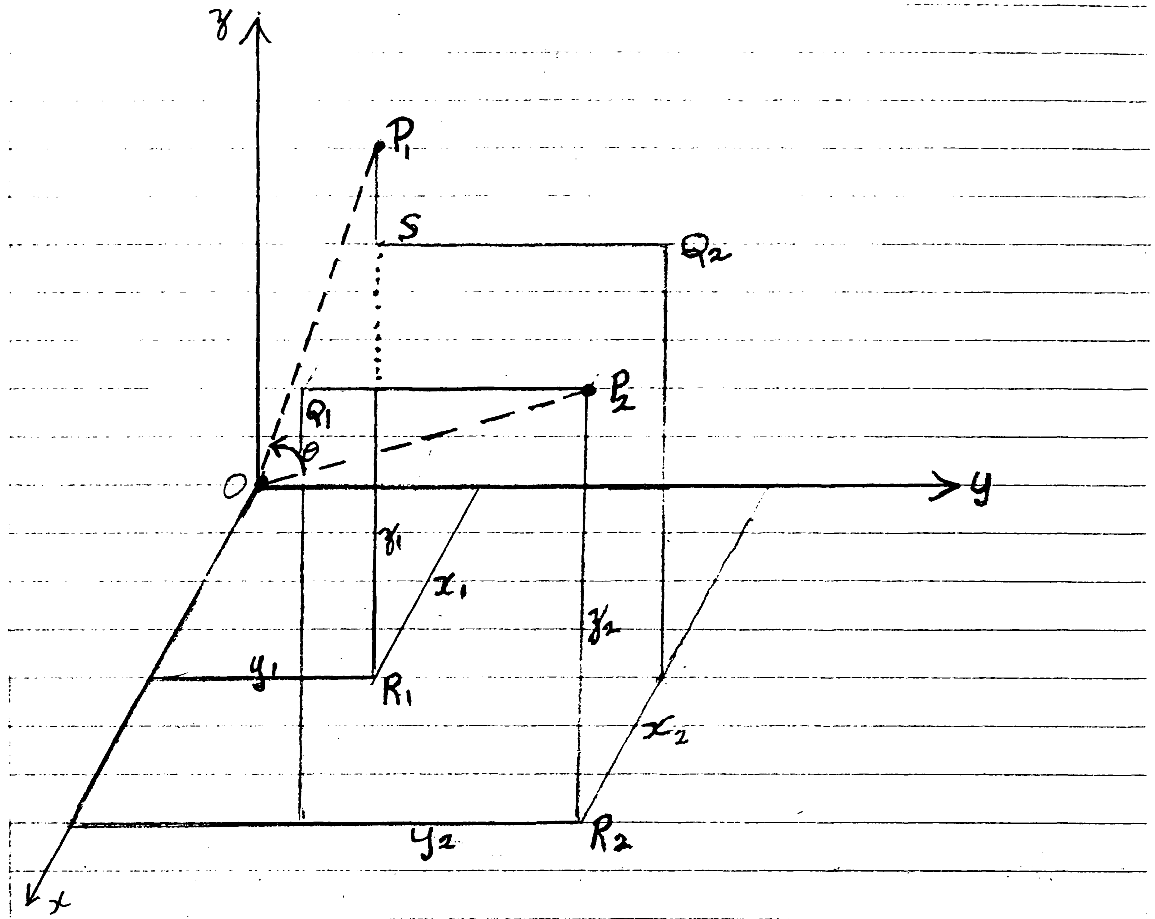

Figure 2. Two vectors $0P_1$ and $0P_2$ in 3-space.

$0P_2$ meeting $0P_2$ at T, labelling $|0T| = t$ as in Figure 3.

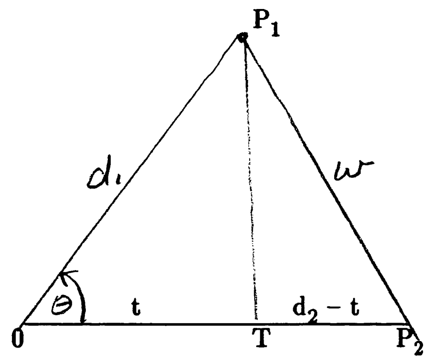

Figure 3. Triangle $P_1 0 P_2$ from Figure 2.

Then

$$\cos\theta = t/d_1 \tag{5}$$

and

$$|P_1T|^2 = d_1^2 - t^2 = w^2 - (d_2 - t^2)$$

which yields

$$t = (d_1^2 + d_2^2 - w^2)/2d_2 \, .$$

Substituting for d_1^2 and d_2^2 from (3) and for w^2 from (4) gives

$$t = (x_1x_2 + y_1y_2 + z_1z_2)/d_2$$

and so

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{d_1d_2}$$

in keeping with (1) for which $\mathbf{r}' = [x_1 \;\; x_2 \;\; x_3]$ and $\mathbf{s}' = [x_2 \;\; y_2 \;\; z_2]$.

ROTATION OF AXES

Rotating axes does not affect the angle between two vectors so it's cosine is unaffected. But co-ordinates of points on the vectors get changed; and yet the formula for the cosine is the same in terms of the new co-ordinates as it is for the old. Illustration of this for the 2-space case is given in terms of Figures 4a and 4b.

In both Figures 4a and 4b (see page) the cartesian axes are labelled X_1 and X_2, and are shown with single arrowheads. After counter-clockwise rotation through an angle β they are labelled X_1' and X_2' with two arrowheads. In both figures OT represents a vector with T having co-ordinates (x_1, x_2)

Rotation of axes through an angle β.

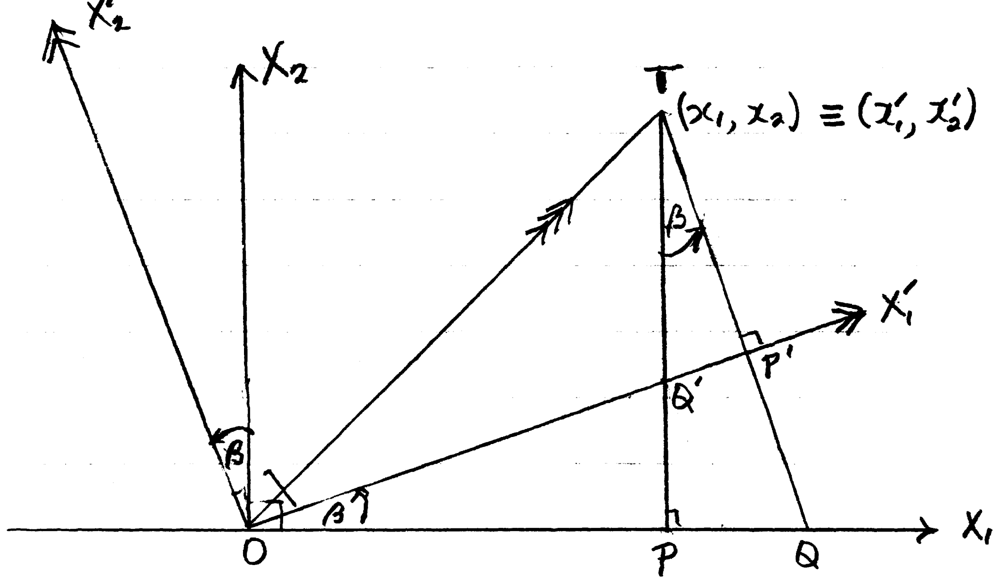

Figure 4a: β < angle TOX_1

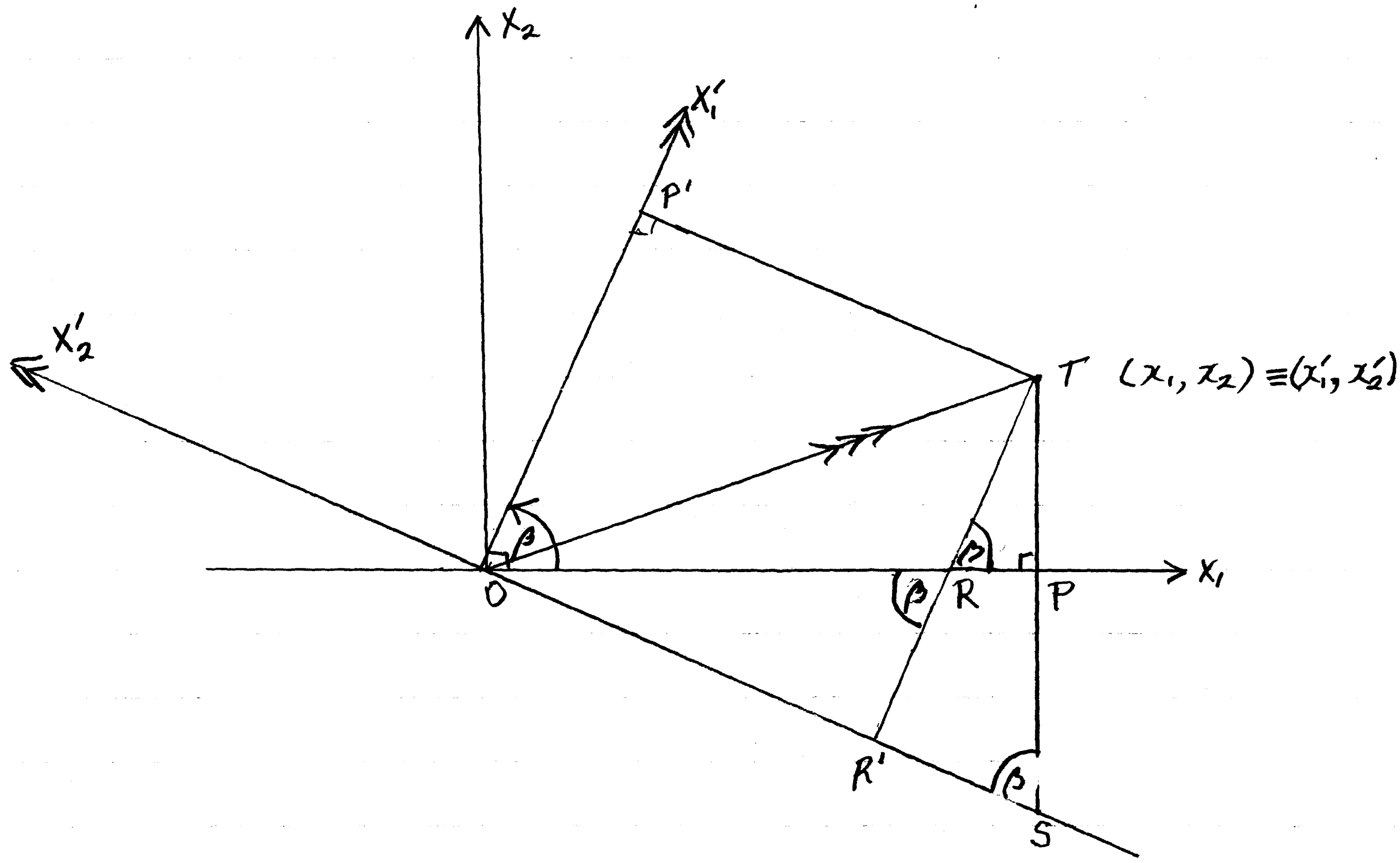

Figure 4b: β > angle TOX_1

and after rotation of axes co-ordinates (x_1', x_2'). The vector is marked with three arrowheads.

In Figure 4a, the angle β is less than the angle between the vector and the X_1 axis; in Figure 4b it is greater. In both figures TP is perpendicular to the X_1 axis and TP′ is perpendicular to the X_1' axis.

Hence

$$x_1 = OP, \quad x_2 = TP, \quad x_1' = OP' \quad \text{and} \quad |x_2'| = TP',$$

the latter arising from the fact that

$$x_2' = TP' \text{ in Figure 4a,} \quad \text{but} \quad x_2' = -TP' \text{ in Figure 4b},$$

where, for example, OP means the distance from O to P.

In contrast, the xs and x′s, being co-ordinates of a point, have length and sign, x_2' being negative in Figure 4b.

We now express x_1' and x_2' in terms of x_1 and x_2, doing so for each figure separately. In Figure 4a

$$x_1' = OP' = OQ' + Q'P' = \frac{OP}{\cos\beta} + TP'\tan\beta = \frac{x_1}{\cos\beta} + x_2'\tan\beta$$

$$= \frac{x_1 + x_2'\sin\beta}{\cos\beta}$$

and

$$x_2' = TP' = TQ - P'Q = \frac{TP}{\cos\beta} - OP'\tan\beta = \frac{x_2}{\cos\beta} - x_1'\tan\beta$$

$$= \frac{x_2 - x_1'\sin\beta}{\cos\beta}.$$

Therefore

$$x_1' = \frac{x_1 + (x_2 - x_1'\sin\beta)\sin\beta/\cos\beta}{\cos\beta}$$

which yields

$$x_1' = x_1\cos\beta + x_2\sin\beta$$

and so

$$x_2' = \frac{x_2 - (x_1\cos\beta + x_2\sin\beta)\sin\beta}{\cos\beta}$$

giving

$$x_2' = x_2\cos\beta - x_1\sin\beta.$$

Clearly

$$d_{x'} = x'^2 + x_2'^2 = (x_1^2 + x_2^2)(\cos^2\beta + \sin^2\beta) + 2x_1x_2(0) = x_1^2 + x_2^2 = d_x \tag{6}$$

as it should.

In Figure 4b, initially using x_2' as just a distance,

$$x_1' = OP' = TR' = TR + RR' = \frac{TP}{\sin\beta} + OP'\cot\beta = \frac{x_2}{\sin\beta} + x_2'\cot\beta$$

$$= \frac{x_2 + x_2'\cos\beta}{\sin\beta}$$

and

$$x_2' = TP' = OR' = OS - R'S = \frac{OP}{\sin\beta} - \frac{TR'}{\tan\beta} = \frac{x_1}{\sin\beta} - \frac{x_1'}{\tan\beta}$$

$$= \frac{x_1 - x_1'\cos\beta}{\sin\beta} .$$

Hence

$$x_1' = \frac{x_2 + (x_1 - x_1'\cos\beta)\cos\beta/\sin\beta}{\sin\beta}$$

giving

$$x_1' = x_1\cos\beta + x_2\sin\beta .$$

Thus

$$x_2' = \frac{x_1 - (x_1\cos\beta + x_2\sin\beta)\cos\beta}{\sin\beta}$$

$$= x_1\sin\beta - x_2\cos\beta .$$

Now because x_2' as a co-ordinate is, in this case, the negative of its length,

$$x_2' = x_2\cos\beta - x_1\sin\beta .$$

Therefore, in both Figures 4a and 4b

$$x_1' = x_1\cos\beta + x_2\sin\beta \qquad \text{and} \qquad x_2' = x_2\cos\beta - x_1\sin\beta . \tag{7}$$

For a second vector, as in Figure 2, (y_1, y_2) and (y_1', y_2') are connected in exactly the same way as are the xs in (7). Therefore, using $s \equiv \sin\beta$ and $c \equiv \cos\beta$,

$$x_1'y_1' + x_2'y_2' = (x_1c + x_2s)(y_1c + y_2s) + (x_2c - x_1s)(y_2c - y_1s)$$

$$= (x_1y_1 + x_2y_2)(c^2 + s^2) + (x_1y_2 + x_2y_1)(cs - cs)$$

$$= x_1y_1 + x_2y_2 .$$

Thus, on also using (6)

$$\cos B = \frac{x_1y_1 + x_2y_2}{d_x d_y} = \frac{x_1'y_1' + x_2'y_2'}{d_{x'}d_{y'}} ,$$

so illustrating that $\cos B$ is the same function of co-ordinates as it is before rotation.

References

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